Fourier Series

Recall in Linear Algebra:

a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ or $\mathbb{C}^n$

usually, $\vec{v}$ use standard basis

$\{\vec{e}_1, \cdots, \vec{e}_n\}$

If we have another set of orthonormal basis

$\{\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n\}$,

we know $\vec{v}$ can be rewritten

as $\vec{v} = \beta_1 \vec{b}_1 + \beta_2 \vec{b}_2 + \cdots + \beta_n \vec{b}_n$

To find each β_i,

we take inner product between $\vec{v}$ and $\vec{b}_i$

$$\vec{v} \cdot \vec{b}_i = \beta_1 \underbrace{\vec{b}_1 \cdot \vec{b}_i}_{0} + \cdots + \beta_i \underbrace{\vec{b}_i \cdot \vec{b}_i}_{1} + \cdots + \beta_n \underbrace{\vec{b}_n \cdot \vec{b}_i}_{0}$$

$$= \beta_i$$

<u>Remark</u> $\vec{a} \cdot \vec{b} = \sum a_i \overline{b_i}$ for complex vector

$$\vec{a} \cdot \vec{b} = \overline{\vec{b} \cdot \vec{a}}.$$

$R[-\pi, \pi]$ = set of Riemann Integrable function on $[-\pi, \pi]$.

Inner Product

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx$$

$$S := \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx \right\}_{n=1}^{\infty}$$

HW 1 Q3 $\Rightarrow$ This set is orthonormal.

For a $f \in R[-\pi, \pi]$, we "change the basis" by:

$$\underbrace{\left\langle f, \frac{1}{\sqrt{2\pi}} \right\rangle \cdot \frac{1}{\sqrt{2\pi}}}_{a_0} + \sum_{n=1}^{\infty} \underbrace{\left\langle f, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle \cdot \frac{1}{\sqrt{\pi}}}_{a_n} \cos nx$$

$$+ \sum_{n=1}^{\infty} \underbrace{\left\langle f, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle \cdot \frac{1}{\sqrt{\pi}}}_{b_n} \sin nx$$

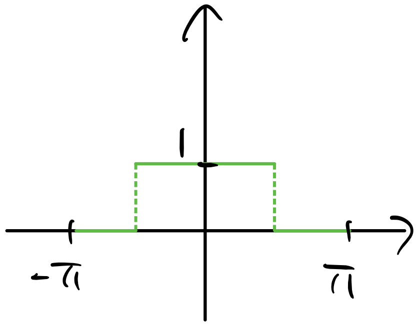
$$= \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx}_{\text{Fourier Series.}}$$

Remark Convergence is Fourier Series is not guaranteed.

But the convergence of Fourier Series is assumed in this Course.

Example 1

$$f(x) = \begin{cases} 1, & x \in [-\pi/2, \pi/2] \\ 0, & \text{else.} \end{cases}$$



$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\,dx = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nx\,dx$$

$$= \frac{1}{\pi}\frac{1}{n}\left[\sin nx\right]_{-\pi/2}^{\pi/2}$$

n even $\Rightarrow$ $a_n = 0$

n odd $\Rightarrow$ $a_n = \frac{2}{n\pi}(-1)^{(n-1)/2}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin nx\,dx$$

$$= \frac{1}{\pi}\frac{1}{n}\left[-\cos nx\right]_{-\pi/2}^{\pi/2}$$

$$= 0$$

$$f \sim \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2}{(2n+1)\pi}(-1)^n \cos(2n+1)x$$

$g \in R[-L, L]$:

$$g(x) = \begin{cases} 1 & , x \in [-L/2, L/2] \\ 0 & , \text{else.} \end{cases}$$

$$h(x) = g\left(\frac{L}{\pi} x\right) \text{ on } [-\pi, \pi].$$
$$= f(x)$$

$$\therefore h \sim \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2}{(2n+1)\pi} (-1)^n \cos(2n+1)x$$

$$g(y) = h\left(\frac{\pi}{L} y\right)$$

$$\sim \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2}{(2n+1)\pi} (-1)^n \cos(2n+1)\frac{\pi}{L} y$$

Or we can consider the set:

$$\left\{ \frac{1}{\sqrt{2L}}, \frac{1}{\sqrt{L}} \sin\left(\frac{n\pi}{L} x\right), \frac{1}{\sqrt{L}} \cos\left(\frac{n\pi}{L} x\right) \right\}_{n=1}^{\infty}$$

The Fourier Series:

$$\left\langle g, \frac{1}{\sqrt{2L}} \right\rangle \cdot \frac{1}{\sqrt{2L}} + \left\langle g, \frac{1}{\sqrt{L}} \sin\left(\frac{n\pi}{L} x\right) \right\rangle \frac{1}{\sqrt{L}} \sin\left(\frac{n\pi}{L} x\right)$$
$$+ \left\langle g, \frac{1}{\sqrt{L}} \cos\left(\frac{n\pi}{L} x\right) \right\rangle \cdot \frac{1}{\sqrt{L}} \cos\left(\frac{n\pi}{L} x\right)$$

In this case,

$$\langle f, g \rangle = \int_{-L}^{L} f \, \bar{g}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} g(x)\,dx = \frac{1}{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} g(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{1}{L}\int_{-L/2}^{L/2}\cos\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{1}{L}\cdot\frac{L}{n\pi}\left[\sin\left(\frac{n\pi}{L}x\right)\right]_{-L/2}^{L/2}$$

$$= \frac{2}{n\pi}\left(\sin\frac{n\pi}{2}\right)$$

n even $\Rightarrow$ $a_n = 0$

n odd $\Rightarrow$ $a_n = \frac{2}{n\pi}(-1)^{(n-1)/2}$

$$b_n = \int_{-L}^{L} g(x)\sin\left(\frac{n\pi}{L}x\right)dx = 0$$

$$g \sim \frac{1}{2} + \sum_{n=0}^{\infty}\frac{2}{(2n+1)\pi}(-1)^n\cos\left(\frac{2n+1}{L}\pi x\right)$$

<u>Remark</u>:

- If f is even on $[-\pi,\pi]$, $f\sin nx$ is odd and so $\int_{-\pi}^{\pi} f\sin nx\,dx = 0$ $\forall n\in\mathbb{N}$

- If f is odd on $[-\pi,\pi]$, $f\cos nx$ is odd and so $\int_{-\pi}^{\pi} f\cos nx\,dx = 0$ $\forall n\in\mathbb{N}$

Analytic Spectral Method

For a Linear DE:

$$L u = s$$

L: Linear Differential Operator

(e.g. $\frac{d^2}{dx^2}$, $\frac{d}{dx}$, some linear combinations etc).

Rewrite u and s

in terms of basis function $\{\phi_n\}_{n\in\mathbb{N}}$.

i.e. $u = \sum_{n=1}^{N} A_n \phi_n$, $s \approx \sum_{n=1}^{N} B_n \phi_n$

and $L\phi_n = \sum_{m=1}^{N} \lambda_m^n \phi_m$

Then $Lu = s$

$$\Rightarrow \sum_{n=1}^{N} A_n \sum_{m=1}^{N} \lambda_m^n \phi_m = \sum_{n=1}^{N} B_n \phi_n$$

$$\Rightarrow \sum_{m=1}^{N} \left(\sum_{n=1}^{N} A_n \lambda_m^n \right) \phi_m = \sum_{m=1}^{N} B_m \phi_m$$

We know s $\Rightarrow$ we know B_m,

we know L and ϕ_n $\Rightarrow$ we know λ_m^n,

Hence we can solve A_n by comparing coef.

Example 2 (Wave Equation)

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}, & (x,t) \in (-\pi,\pi) \times (0,\infty) \\ 0 = u(-\pi,t) = u(\pi,t), & t > 0 \\ u(x,0) = f, \ \dfrac{\partial u}{\partial t}(x,0) = g, & -\pi < x < \pi \end{cases}$$

Let $u(x,t) = X(x)\,T(t)$.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \underbrace{\frac{X''(x)}{X(x)}}_{\text{Independent on } t} = \underbrace{\frac{T''(t)}{T(t)}}_{\text{Independent on } x}$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \lambda, \ \lambda \text{ constant.}$$

$$X''(x) = \lambda X(x)$$

$$\Rightarrow X(x) = \begin{cases} \alpha_1 e^{\sqrt{\lambda}x} + \alpha_2 e^{-\sqrt{\lambda}x}, & \lambda > 0 \\ \alpha_1 x + \alpha_2, & \lambda = 0 \\ \alpha_1 \cos\sqrt{-\lambda}x + \alpha_2 \sin\sqrt{-\lambda}x, & \lambda < 0 \end{cases}$$

For $\lambda \geq 0$,

B. C. $\Rightarrow \alpha_1 = \alpha_2 = 0$ or $T(t) \equiv 0$.

$\Rightarrow u(x,t) = X(x)\,T(t) \equiv 0$.

For $\lambda < 0$,

B. C. $\Rightarrow$ $n = \sqrt{-\lambda} \in \mathbb{N}$,

$\alpha_1 = 0$

$$T''(t) = \lambda T(t) = -n^2 T(t)$$

$$\Rightarrow T(t) = \beta_1 \cos nt + \beta_2 \sin nt$$

$$\therefore \quad u_n(x,t) = X_n(x)\,T_n(t)$$
$$= \left(A_n \cos nt + B_n \sin nt\right) \sin nx$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$
$$= \sum_{n=1}^{\infty} \left(A_n \cos nt + B_n \sin nt\right) \sin nx$$

$$u(x,0) = f \Rightarrow \sum_{n=1}^{\infty} A_n \sin nx = f$$

$$u_t(x,0) = g \Rightarrow \sum_{n=1}^{\infty} n B_n \sin nx = g$$

Solve A_n, B_n using Fourier Series.

Hint for HW 1 Q 6.

$$\begin{cases} \dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2}, & 0 < x < \pi,\ t > 0. \\ \dfrac{\partial u}{\partial x}(t,0) = 0 = \dfrac{\partial u}{\partial x}(t,\pi), & t > 0 \\ u(0,x) = x, & 0 \le x \le \pi \end{cases}$$

Consider a even extension on x-coordinate.

$$v(t,x) = \begin{cases} u(t,x), & x \ge 0 \\ u(t,-x), & x < 0. \end{cases}$$

Then the PDE becomes:

$$\begin{cases} \dfrac{\partial v}{\partial t} = 4\dfrac{\partial^2 v}{\partial t^2}, & -\pi < x < \pi,\ t > 0 \\ 0 = \dfrac{\partial v}{\partial x}(t,-\pi) = \dfrac{\partial v}{\partial x}(t,\pi), & t > 0 \\ v(0,x) = \begin{cases} x, & 0 \le x \le \pi \\ -x, & -\pi \le x < 0 \end{cases} \end{cases}$$

Condition $\dfrac{\partial u}{\partial x}(t,0) = 0$ is automatically satisfy:

$$\frac{\partial v}{\partial x}(t,0) = \lim_{h \to 0^+} \frac{v(t,h) - v(t,-h)}{2h}$$

$$= 0.$$

Exercise

(a) Compute Fourier Series:

$$f(x) = \begin{cases} 1, & x \in [\pi, 2\pi] \\ -1, & x \in [0, \pi). \end{cases}$$

(b). Solve using Spectral Method:

$$y'' + 4y = \sin 4x + \cos 6x$$

on $[-\pi, \pi]$

Exercise Solution:

(a). $f(x) = \begin{cases} 1, & x \in [\pi, 2\pi] \\ -1, & x \in [0, \pi) \end{cases}$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \,dx$$

$$= \frac{1}{\pi}\left[\int_\pi^{2\pi} \cos nx \,dx - \int_0^\pi \cos nx \,dx \right]$$

$$= \frac{1}{\pi}\frac{1}{n}\left[[\sin nx]_\pi^{2\pi} - [\sin nx]_0^\pi \right] = 0$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \,dx$$

$$= \frac{1}{\pi}\left[\int_\pi^{2\pi} \sin nx \,dx - \int_0^\pi \sin nx \,dx \right]$$

$$= \frac{1}{\pi}\frac{1}{-n}\left[[\cos nx]_\pi^{2\pi} - [\cos nx]_0^\pi \right]$$

$$= \frac{-1}{n\pi}\left[\underbrace{\cos 2n\pi}_{1} - \underbrace{\cos n\pi}_{(-1)^n} - \underbrace{\cos n\pi}_{(-1)^n} + \underbrace{\cos 0}_{1} \right]$$

$$= \frac{2}{n\pi}\left((-1)^n - 1 \right)$$

(c)

Or Consider:

$$f_2(x) := f(x+\pi) = \begin{cases} 1, & x \in [0, \pi] \\ -1, & x \in [-\pi, 0) \end{cases}$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_2 = 0, \quad A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_2 \cos nx = 0$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_2(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \sin nx \, dx - \int_{-\pi}^{0} \sin nx \, dx\right]$$

$$= \frac{2}{\pi} \cdot \frac{1}{n}\left[-\cos nx\right]_0^{\pi}$$

$$= \frac{2}{n\pi}\left(1 - (-1)^n\right)$$

$$f_2 \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 - (-1)^n\right)\sin nx$$

$$f \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 - (-1)^n\right)\sin(n(x-\pi))$$

$$= \sum_{n=1}^{n} \frac{2}{n\pi}\left((-1)^n - 1\right)\sin nx$$

(b). $y'' + 4y = \sin 4x + \cos 6x$, on $[-\pi, \pi]$

$$y \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$y'' \sim \sum_{n=1}^{\infty} (-n^2)(a_n \cos nx + b_n \sin nx)$$

$$y'' + 4y \sim a_0 + \sum_{n=1}^{\infty} (-n^2+4)(a_n \cos nx + b_n \sin nx).$$

Comparing Coef: $(-4^2+4)\, b_4 = 1$

$$b_4 = \frac{1}{-12}$$

$$(-6^2+4)\, a_6 = 1$$

$$a_6 = \frac{1}{-32}$$

We also need to consider the terms where $(-n^2+4) = 0$.

$$n = 2 \text{ or } -2$$

$$\therefore\ y(x) = \alpha_1 \cos 2x + \alpha_2 \sin 2x - \frac{1}{12}\sin 4x - \frac{1}{32}\cos 6x.$$

Or like what we did in Integrating Factor:

$y_1'' + 4y_1 = 0$, $y_2'' + 4y_2 = \sin 4x + \cos 6x$

$$\Rightarrow\ y_1(x) = \alpha_1 \cos 2x + \alpha_2 \sin 2x.$$